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A METHOD OF GLOBAL RANDOM SEARCH IN INVERSE PROBLEMS WITH APPLICATION TO THE PROBLEM OF RECOGNIZING THE SHAPE OF A DEFECT†

M. A. Sumbatyan

Rostov-on-Don

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For solving ill-posed inverse problems, the method based on the stochastic approach to the minimization of the smoothing discrepancy functional is used. The method is applied both to linear and non-linear problems and, as opposed to classical methods, it is not required to invert the operator of the direct problem. As an application the problem of reconstructing the shape of a scatterer from the wave field diffracted by it is considered.

1. Many problems in mathematical physics reduce to solving the operator equation

$$Ax = f \tag{1.1}$$

where A is a completely continuous operator acting in some Hilbert space H. The problem of finding x, when the right-hand side f is known, is therefore an inverse problem. It is known [1] that Eq. (1.1) is considered to be a Tikhonov ill-posed problem in the sense that a large change in the solution x may correspond to a small change in the right-hand side. This leads to instability of the basic classical numerical methods when they are applied to Eq. (1.1).

To overcome these difficulties, special numerical methods [1] have been developed. One way or another most of them are based on the ideas of regularizing Eq. (1.1) by means of an appropriate small perturbation of the operator A.

In this paper a method which is natural from the physical point of view is proposed. If approximations x_n are selected randomly, it is possible to evaluate their closeness to the exact solution x^*, from the closeness of the left- and right-hand sides, that is, from the smallness of the discrepancy functional.

$$\psi(x_n) = \|Ax_n - f\|^2 \tag{1.2}$$

But it is known [1] that solving the ill-posed equation (1.1) cannot be reduced to minimizing the functional $\psi(x)$ (1.2). Instead, it is necessary to consider the smoothing functional [1]

$$\psi_\epsilon(x) = \|Ax - f\|^2 + \epsilon R(x), \quad 0 < \epsilon \ll 1 \tag{1.3}$$

To minimize functional (1.3) we will use the method of global random search (see [2, algorithm 5.1.3]). Unlike direct random search, it has important properties which enable the process of finding a good approximation x_n to be accelerated. This is valid because

1. random sampling of values x_n in the neighbourhood of the points X, for which the values of $\psi_\epsilon(X)$ are smaller, happens more frequently than that in the neighbourhood of the points Y, for which the values of $\psi_\epsilon(Y)$ are larger, and

2. domains, in which random values x_n are chosen, are gradually contracted to the small neighbourhoods of points that have the least values of $\psi_\epsilon(x)$.

This algorithm has been tested [2] using the examples of minimizing finite-dimensional functions of a small number of variables. Here we use it to solve functional equations in Hilbert space.

We will first consider the integral equation of the first kind

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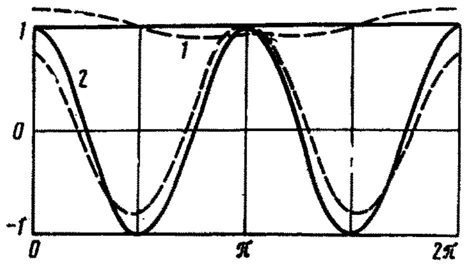

FIG. 1.

$$Ax = \int_0^{2\pi} \cos[\rho\cos(\varphi - \theta)]\, x(\theta)\, d\theta = f(\varphi), \quad 0 \leqslant \varphi < 2\pi, \quad \rho = \text{const} \tag{1.4}$$

as a model example. Its kernel is infinitely differentiable, which implies that this equation is extremely ill-posed.

We will find its solution in the form of a Fourier series with a finite number of terms

$$x(\theta) = \sum_{m=0}^{M} a_m \cos m\theta, \quad 0 \leqslant \theta < 2\pi \tag{1.5}$$

For simplicity, we consider here the case when the right-hand side $f(x)$ (as well as the kernel) is an even function of φ. In this approach the minimization of the functional (1.3), based on the method of global random search described above, implies random sampling of sets of real numbers $(a_0, a_1, \ldots, a_m)_n$. It is proposed to choose the regularizing functional in the form [1]

$$R(x) = \| x \|_{W_2^2(0,\, 2\pi)} \tag{1.6}$$

where W_2^2 is Sobolev space, and the basic space is $H = L_2(0, 2\pi)$.

The results of the application of the proposed algorithm (the dashed lines) for two right-hand sides, which are $f_1(\varphi) = 2\pi J_0(\rho)$ and $f_2(\varphi) = -2\pi J_2(\rho)\cos 2\varphi$ (curves 1 and 2), are shown in Fig. 1. For these two cases the explicit solutions have the forms $x_1(\vartheta) = 1$ and $x_2(\vartheta) = \cos 2\vartheta$, respectively. They are represented by continuous lines. The parameters take the values $\rho = 1$, $M = 4$, $\epsilon = 10^{-2}$–10^{-4}. The number of random samples of the solution defining the number of calculations of the direct operator A was equal to $N = 300$.

2. The method given in Sec. 1 is especially effective for problems characterized by a large number of local minima of functional (1.3), as well as by large values of the gradients of the functional. In such problems it is practically impossible to find the global minimum by regular methods. As a detailed analysis indicates, the problems of recognizing an object from the wave field scattered by it are distinguished by these properties.

When investigating the problem of recognizing the shape of a defect, we will restrict ourselves, for simplicity, to the two-dimensional case and to the simplest model of an acoustic medium described by a single Helmholtz equation. Then it is possible to reduce the problem of a system of two non-linear integral equations [3]

$$\left| \int_l e^{ik(q\cdot x)} g(x)\, ds_x \right| = f(\alpha), \quad 0 \leqslant \alpha < 2\pi \tag{2.1}$$

$$\int_l H_0^{(1)}(k|x - y|) g(x)\, ds_x = e^{ik(q\cdot y)}, \quad y \in l \tag{2.2}$$

$$q = -\{\cos\alpha, \ \sin\alpha\}$$

Here k is the wave number and $H_0^{(1)}$ is the Hankel function. In Eqs (2.1) and (2.2) the unknowns are the function $g(x)$, connected with the normal derivative of the velocity, and the function $x \in l$, which defines the position of the boundary contour l (the latter may be specified, for example, in some parametric form).

System (2.1), (2.2) holds for the case when the amplitude of the back-scattering from the object is known for the whole range of variation of the scanning angle $\alpha \in (0, 2\pi)$. This method of scanning, when the directions of propagation of the incident wave and the reflected wave coincide, corresponds to the echo-method, widely used in ultrasonic testing. In this method the same ultrasonic sensor device serves both as the emitter and receiver. The amplitude of the reflected signal $f(\alpha)$ is therefore known, but the phase, as a rule, is unknown. It is assumed that similar measurements may be carried out in principle for any angle of incidence $\alpha \in (0, 2\pi)$.

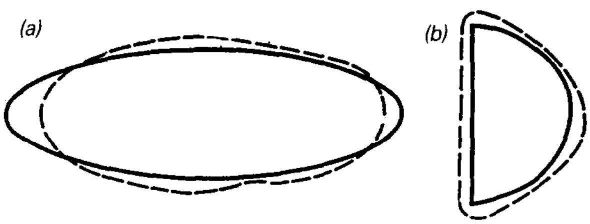

FIG. 2.

The direct problem of diffraction consists in calculating the function $f(\alpha)$ along the known contour l. To do this it is necessary, first, to solve the boundary integral equation (2.2) for the function $g(x)$, $x \in l$, and then calculate the quadrature (2.1). It is obvious that the corresponding operator Ax in (1.1) is non-linear. Calculating the direct action of a non-linear operator consequently reduces to solving one linear equation and one quadrature. This situation is typical for non-linear inverse problems. Unlike the above case the inverse operator A^{-1} (if it exists) is extremely non-linear. When using regular methods to invert Eq. (1.1) it would be necessary to calculate the Frêchet derivative for the operator A, which is quite difficult to do. The method described in Sec. 1 requires calculating only the direct operator, which, as noted above, involves linear operations only.

The results of reconstructing two objects, an ellipse with a ratio of its semi-axes of 3 : 1 and a semicircle with a diameter of 10, from the amplitude of circular back-scattering are in Fig. 2. The wave length was taken as $\lambda = 2\pi$. As in the model example, the parametric representation of the contour l was specified in the form of a finite truncation of a Fourier series with $M = 4$. The true contour is shown by the continuous line and the result of using the algorithm described is shown by the dashed line. The typical size of the objects was (1–1.5)λ. The average time needed for calculation on a PC/AT 286/287 was 20–25 min.

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